The Molecular-Kinetic Basis of Hydrodynamic Equations

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THE MOLECULAR-KINETIC BASIS OF HYDRODYNAMIC EQUATIONS

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Derivation of hydrodynamic equations (taking into account internal friction and thermal conductivity equations) from the standpoint of the hypothesis of elastic intercollision of molecules is a most arduous task. The hypothesis of mutual interaction of molecules at a distance permits a relatively easy solution of the problem, especially on assuming that the force of molecular interaction is repelling and inversely proportional to the fifth power of the distance between interacting molecules. Such a law may be conveniently adopted also because many data of the molecular kinetic theory, derived on the basis of this hypothesis, tally satisfactorily with experimental results. There can be some doubt relative to the accuracy of some numerical coefficients, but none as to that of the analytical correlations. Subsequently, after Maxwell had utilized the above-cited law of interaction in solving a number of problems pertaining to the molecular kinetic theory, Langevin confirmed the calculations on the basis of any law of interaction force between the molecules. Maxwell's hypothesis relative to the interaction of molecules on their coming together was not fortuitous for two reasons: first, Maxwell was aware that the laws of elastic collisions can be approximated with a sufficient degree of correctness by the law of repulsion between molecules having come together; in this respect it is but necessary to select a corresponding rate of force attenuation with distance. Exact concordance of calculations, which then cease to be approximate, takes place only when alteration of force with distance, corresponds not to the fifth power but to a power equal to ∞. Experiments have shown however that the fifth power sufficiently approximates the results of those obtained on the basis of the elastic spheres.

The second consideration, which precludes the assumption that Maxwell's hypothesis is a fortuitous one, is inherent in the modern concepts of the nature of molecular forces. At present we must divide the zone of molecular interaction into two regions: (1) the region in which repulsing forces are acting, and (2) the region in which the acting forces are those of attraction. The latter are made apparent only when an association of molecules or a chemical process obtains.

Consequently if the condition of the system is such that neither an association of molecules nor a chemical process takes place therein, in effecting molecular kinetic calculations of phenomena which occur in the system, the use of Maxwell's concepts is permissible.

I. SOME AUXILIARY FORMULAS

1. Let us consider a medium of identical molecules located in a field of any forces whatever. Let $\vec{W_1}$ be the vector of velocity of any one molecule. Let us resolve this vector into two: one characterizing the harmonious (orderly) motion of the molecules, and the other their disharmonious (disorderly) motion.

Thus, we have

$$\overrightarrow{W_1} = \overrightarrow{W_{11}} + \overrightarrow{u_1} \qquad (1)$$

Both vectors $\overrightarrow{W_{11}}$ and $\overrightarrow{u_1}$ we will consider as dependent on coordinates (x_1, x_2, x_3) and the time t.

2. Let, at the point (x_1, x_2, x_3) of our medium at the moment t, the unit of volume contains molecules having the velocity $\overrightarrow{W_1}$, in a number equal to

$$dN_1 = f(W_{1x_1}, W_{1x_2}, W_{1x_3})\, dW_{1x_1}\, dW_{1x_2}\, dW_{1x_3}.$$

Let at another point, at a distance from the first equal to the zone of molecular interaction, the number of molecules having the velocity $\overrightarrow{W_2}$, be equal to:

$$dN_2 = f(W_{2x_1}, W_{2x_2}, W_{2x_3})\, dW_{2x_1}\, dW_{2x_2}\, dW_{2x_3}.$$

The molecules of the first of these two groups have a velocity relative to those of the second group, which is given by the equation:

$$\mathfrak{B}^2 = (W_{1x_2} - W_{2x_2})^2 + (W_{1x_2} - W_{2x_2})^2 + (W_{1x_3} - W_{2x_3})^2.$$

approaches

The number of convergencies involving the molecules of these groups, which occur within the unit of volume, over the length of time dt, will be equal to:

$$\mathfrak{B}\, b\, db\ \ d\psi\ \ dt\ \ dN_1\, dN_2$$

Here we use b to denote the distance of the first molecule from the direction line of relative velocity; ψ denotes the two-plane angle formed by the plane passing through the line connecting the converged molecules and the vector of relative velocity, with the plane passing through axis x_1 and the vector of relative velocity. Thus the line of direction of the relative velocity will be the secant of the above indicated planes (see figure).

3. Let us denote by Q the quantitative expression of some quality transferred, for example, by the first molecule. This quantity is a function of the components of velocity vector $\vec{W_1}$. After convergence of molecules of the first and second kind, quantity Q, generally speaking, will change and acquire a value equal to Q'. As a result of all the convergencies of molecules during the length of time dt within the unit of volume, quantity Q will change to the extent:

$$(Q'-Q)\,\mathcal{B}\,b\,db\,d\psi\,dt\,dN_1\,dN_2. \qquad (2)$$

On integrating expression (2) with respect to ψ from 0 to 2π, with respect to b from 0 to ∞, and with respect to velocities from $-\infty$ to $+\infty$, we obtain the increase of the sum of quantities Q for the unit of volume during the length of time dt, which takes place as a result of convergency of molecules of the first and second kind. However, integration of expression (2) with respect to velocities can not be effected, since we do not know the function $f(W_{x_1}, W_{x_2}, W_{x_3})$, but for our purpose it is sufficient to effect this integration using mean values, that is by utilizing the integral of the form

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$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(W_{1x_1}, W_{1x_2}, W_{1x_3}, W_{2x_1}, W_{2x_2}, W_{2x_3})\, dN_1\, dN_2 = N_1 N_2 \bar{\varphi}. \quad (3)$$

4. Under the influence of central forces the converged molecules will move in such a manner that one of them, for example the second, can be considered as being a stationary center, exercising a force the potential of which is equal to:

$$\Phi = \frac{m_1 + m_2}{m_2} f(r).$$

If the polar axis direction is reversed relative to the initial relative velocity of molecule m_1, and the polar angle φ considered positive on that side of the polar axis where the molecule m_1 was located at the initial moment, then on the basis of the law of conservation of kinetic forces and the law of constant areas, we can write the following two equations:

$$\frac{dr^2 + r^2 d\varphi^2}{dt^2} + 2\frac{m_1 + m_2}{m_1 m_2} f(r) = \mathfrak{B}^2 \text{ and } r^2 \frac{d\varphi}{dt} = b\mathfrak{B}.$$

From these two equations it follows that

$$\frac{d\varphi}{dr} = \frac{b\mathfrak{B}}{\pm\sqrt{\left[\mathfrak{B}^2 - 2\frac{m_1 + m_2}{m_1 m_2} f(r)\right] r^4 - b^2 \mathfrak{B}^2 r^2}}$$

wherefrom we have:

$$\varphi = \int_{r_0}^{\infty} \frac{b\mathfrak{B}\, dr}{\sqrt{\left[\mathfrak{B}^2 - 2\frac{m_1 + m_2}{m_1 m_2} f(r)\right] r^4 - b^2 \mathfrak{B}^2 r^3}}. \quad (4)$$

Here r_0 is used to denote the minimum distance within which
approach
two colliding molecules are converged.

Expression (4) can be substantially simplified by taking into consideration the following conditions:

$$m_1 = m_2, \quad \alpha^4 = \frac{b^4 \mathcal{B}^4}{2\kappa m}, \quad f(r) = \kappa m_1 m_2 \varphi(r), \quad r = \frac{b}{y}.$$

In this case we have

$$\varphi = \theta = \int_0^{y_0} \frac{dy}{\sqrt{1 - y^2 - \frac{b^4}{\alpha^4} 2\varphi\left(\frac{b}{y}\right)}}. \qquad (4a)$$

If

$$\varphi(r) = \frac{1}{s} \frac{1}{r^s},$$

the expression (4a) can be written as follows

$$\theta = \int_0^{y_0} \frac{dy}{\sqrt{1 - y^2 - \frac{2y^s}{s\alpha^4 b^{s-4}}}}. \qquad (4b)$$

The integral (4b) is most readily obtained when $s = 4$. In that instance it is reduced to an absolute elliptical integral of the first kind. Values of angle θ are needed for integration with respect to b of expression (2).

In accordance with the foregoing denotations we have:

$$b\,db = \frac{\sqrt{2\kappa m}}{\mathcal{D}} \alpha\, d\alpha.$$

Now expression (2) can be written as follows:

$$(Q'-Q)\sqrt{2\varkappa m}\;\alpha\, d\alpha\, d\psi\, dt\, dN_1\, dN_2. \qquad (2a)$$

5. Problem A

Maxwell's law of molecular interaction requires determination of the integral

$$\int_0^{2\pi} d\psi \int_0^{\infty} \sqrt{2\varkappa m}\;\alpha\, d\alpha \left(W'^{2}_{1x_1} - W^{2}_{1x_1}\right).$$

Solution:

Consider the ~~equation~~ identity

$$W'_{1x_1} = \frac{m_1 W'_{1x_1} + m_2 W'_{2x_2}}{m_1+m_2} + \frac{m_2}{m_1+m_2}\left(W'_{1x_1} - W'_{2x_2}\right).$$

In view of the law of ~~impulse~~ of momentum conservation, we have

$$m_1 W_{1x_1} + m_2 W_{2x_1} = m_1 W_{1x_1} + m_2 W_{2x_2};$$

in addition, the following equation holds:

$$\mathfrak{B}' = \mathfrak{B}$$

Let us denote by A the angle formed by vector $\overrightarrow{\mathfrak{B}'}$ with axis x_1; we have then

$$W'_{1x_1} - W'_{2x_1} = \mathfrak{B}\cos A.$$

In all of the above written expressions the index (') relates to quantities which characterize the condition of the molecules after their convergence.

Taking into consideration all of the foregoing, we have the following expression

$$W'_{1x_1} = \frac{m_1 W_{1x_1} + m_2 W_{2x_1}}{m_1 + m_2} + \frac{m_2}{m_1 + m_2} \mathcal{B} \cos A, \qquad (5)$$

but

$$\cos A = -\cos 2\theta \cos \alpha_0 - \sin 2\theta \sin \alpha_0 \cos \psi. \qquad (5a)$$

Here α_0 is used to denote the angle formed by the vector of relative velocity with the x_1 axis.

On combining expressions (5) and (5a) we have:

$$W'_{1x_1} = W_{1x_1} + \frac{m_2}{m + m_2}\Big[2(W_{2x_2} - W_{1x_1}) \cos^2\theta -$$

$$-\left\{\sqrt{(W_{1x_2} - W_{2x_2})^2 + (W_{1x_3} - W_{2x_3})^2}\right\} \sin 2\theta \cos \psi\Big]. \qquad (6)$$

On transformation, by means of equation (6), of the expression within the integration sign of the proposed problem, we have the following result

$$\sqrt{2\varkappa m}\int_0^{2\pi} d\psi\int_0^{\infty}\alpha\, d\alpha\,(W'^2_{1\varkappa_2}-W^2_{1\varkappa_1})=\sqrt{\frac{\varkappa m}{2}}\{I_1(s)(W^2_{2\varkappa_1}-W^2_{1\varkappa_1})+$$

$$+\frac{1}{2}I_2(s)[(W_{1\varkappa_2}-W_{2\varkappa_2})^2+(W_{1\varkappa_3}-W_{2\varkappa_3})^2-2(W_{1\varkappa_1}-W_{2\varkappa_1})^2]\}. \quad (7)$$

Here $I_1(s)$ and $I_2(s)$ are used to denote the following integrals

$$I_1(s)=4\pi\int_0^{\infty}\alpha\, d\alpha\cos^2\theta, \quad I^2(s)=\pi\int_0^{\infty}\alpha\, d\alpha\sin^2 2\theta.$$

6. Problem B

In connection with the same interaction, determine the integral

$$\sqrt{2\varkappa m}\int_0^{2\pi} d\psi\int_0^{\infty}\alpha\, d\alpha\,(W'_{1\varkappa_1}W'_{1\varkappa_2}-W_{1\varkappa_1}W_{1\varkappa_2}).$$

Solution

By means of considerations analogous to those presented in the foregoing problem, the following equation can be established

$$W'_{1\varkappa_2}=W_{1\varkappa_2}+\frac{m_2}{m_1+m_2}\{2(W_{2\varkappa_2}-W_{1\varkappa_2})\cos^2\theta-$$

$$-\sqrt{(W_{1\varkappa_1}-W_{1\varkappa_2})^2+(W_{1\varkappa_3}-W_{2\varkappa_3})^2}\sin 2\theta\cos(\psi-\psi_{12})\}. \quad (8)$$

Here ψ_{12} is used to denote the two-plane angle formed by the planes passing through the line of relative velocity on convergence of the molecules and through axes x_1 and x_2.

We transform by means of equations (6) and (8) the expression within the sign of integration of the proposed problem. As a result we have

$$\sqrt{2\kappa m}\int_0^{2\pi} d\psi \int_0^{\infty} d\alpha\, \alpha \left(W'_{1x_1} W'_{1x_2} - W_{1x_1} W_{1x_2}\right) =$$

$$= \sqrt{\frac{\kappa m}{2}} \left\{ I_1(s)\, W_{2x_1} W_{2x_2} - W_{1x_1} W_{1x_2}\right) -$$

$$- \frac{3}{2} I_2(s)\left(W_{1x_1} - W_{2x_1}\right)\left(W_{1x_2} - W_{2x_2}\right)\right\}. \qquad (9)$$

Expressions (7) and (9) will be subsequently of determinant significance.

II. CONDITIONS OF TRANSITION TO CONTINUUM

1. In the derivation of equations of motion of a viscous fluid Maxwell utilized on the one hand equations of transfer, and on the other, equations (7) and (9), averaged for all molecules contained in the unit of volume. In so doing he considered the transfer velocities of two colliding molecules as being equal. This means that his continuum, which is in motion, has a filamentous structure, that is, the minimum dimensions of the stria correspond to the mean distance between the molecules. Such a structure of the flow can not always be maintained. At sufficiently high velocities such a motion of the continuum may be disrupted at the wall or on

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formation of vortexes. This hypothesis also becomes doubtful in those instances when the dimensions of the flow are of the same order of the mean distance between molecules, which takes place in the motion of rarefied gases.

Therefore transition from a kinetic molecular description of motion of a medium to its continuous description presents certain difficulties; without some more or less plausible assumptions such a transition is impossible. In lieu of the hypothesis of Maxwell we propose the following conditions of transition to continuum.

2. Let us consider an infinitely small volume, containing (so many) ~~[illegible]~~ molecules, that mean values can be calculated without a significant error. Let the components of the transfer velocity of the mass center of this volume be equal to

$$W_{0x_1},\ W_{0x_2},\ W_{0x_3}$$

In order that a molecule of the first kind may collide with a molecule of the second kind within this volume, even when their thermal velocities are equal in magnitude and direction, it is necessary to meet conditions of the following form:

$$W_{22x_1} < W_{0x_1} < W_{11x_1},\quad W_{22x_2} < W_{0x_2} < W_{11x_2},\quad W_{22x_3} < W_{0x_3} < W_{11x_3}.$$

Furthermore, let the harmonious motion be such that gradients of transfer velocity $\overrightarrow{W_0}$ undergo noticeable change along the free path of the molecules. In such a case it is always possible to select for molecules of the first and of the second kind, three such quantities h_1, h_2, h_3, the values of which will permit to formulate equations of the following form

$$W_{11x_i} = W_{0x_i} + h_1 \frac{\partial W_{0x_i}}{\partial x_1} + h_2 \frac{\partial W_{0x_i}}{\partial x_2} + h_3 \frac{\partial W_{0x_i}}{\partial x_3},$$

$$W_{22x_i} = W_{0x_i} - h_1 \frac{\partial W_{0x_i}}{\partial x_1} - h_2 \frac{\partial W_{0x_i}}{\partial x_2} - h_3 \frac{\partial W_{0x_i}}{\partial x_3}. \qquad (10)$$

Quantities h_1, h_2, h_3, generally speaking, change on transition from one pair of molecules to another.

Let us denote by x_1, x_2, x_3 the values of the edges of a rectangular parallelepiped of infinitely small volume, and let us assume that mean values of the quantities h_1, h_2, h_3 satisfy the following condition

$$\frac{\overline{h_1}}{x_1} = \frac{\overline{h_2}}{x_2} = \frac{\overline{h_3}}{x_3}$$

The question arises, upon what may be dependent these ratios of infinitely small values?

Let us denote by $\overline{W_0}$ the mean transfer velocity along the free path of the molecules; then, obviously, the above-given ratios must be some function of the difference $(\overline{W_0} - W_0)$. In addition this argument must be non-dimensional. We shall express this difference in units of sound velocity, that is, we select as the argument the quantity

$$\beta_s = \frac{\overline{W_0} - W_0}{C} .$$

The ratios (11) must also be dependent on a definite congruence between dimensions of flow and length of free path of the molecules. As a criterion of this congruence let us select the value

$$\beta_L = \frac{D}{L} .$$

Here D denotes the parameter characterizing dimensions of flow, and L the length of free path of the molecules. Obviously it is convenient to take as the ultimate argument of function $\frac{\overline{h_i}}{x_i}$ the non-dimensional criterion

$$Re_0 = \frac{(\overline{W_0} - W_0)D}{CL} .$$

Thus we have

$$\frac{\overline{h_i}}{x_1} = f(Re_0) .$$

Determining the form of this function does not yet appear to be possible. This problem must be solved experimentally.

3. Let us introduce into solution (7) the ratios (10); in so doing let us limit ourselves to the first powers of the quantities h_i. Effecting thereafter the averaging for all molecules of the first kind, we have

$$\frac{\delta W^2_{1x_1}}{\delta t} = -2\rho q I_1(s)(\overline{\vec{h}}, \operatorname{grad} W^2_{0x_1}) + \rho q I_2(s)[\bar{u}^2 - 3\bar{u}^2_{x_1}].$$

Here q is used to denote the quantity $\sqrt{\frac{k}{2m}}$, and ρ, the density of the medium. The left hand portion of the equation so obtained represents variation of quantity $W^2_{1x_1}$ as a result of inter-collisions of molecules; it symbolically denotes the integral of problem A.

In calculating this integral, there are taken into consideration the following obvious values of mean quantities

$$\bar{u}^2_{1x_1} = \bar{u}^2_{2x_1}, \quad \bar{u}^2_{1x_2} = \bar{u}^2_{2x_2}, \quad \bar{u}^2_{1x_3} = \bar{u}^2_{2x_3}, \quad \overline{u_{1x_i} u_{2x_i}} = 0,$$

$$\overline{Qu}_{1x_i} = \overline{Qu}_{2x_i} = 0.$$

Let us assume for pressure p, within our medium, a quantity equal to

$$p = \frac{\rho}{3}(\bar{u}^2_{x_1} + \bar{u}^2_{x_2} + \bar{u}^2_{x_3}) = \frac{\rho}{3}\bar{u}^2.$$

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Taking into account this definition of the pressure and also equation (11a), we can write the above-derived expression of $\frac{\delta W_{1x_1}^2}{\delta t}$ as follows

$$\frac{\delta W_{1x_1}^2}{\delta t} = -2\rho q I_1(s) f(Re_0)(\vec{X}, \operatorname{grad} W_{0x_1}^2) + 3q I_2(s)(P - \rho \bar{u}_{x_1}^2). \tag{12}$$

Here $\vec{X}$ denotes the radius-vector having as its components x_1, x_2, x_3.

4. Let us introduce into equation (9) the ratios (10). Then on maintaining the limitations set forth hereinbefore and on averaging for all molecules of the first kind, we can write for the integral of problem B the following expression

$$\frac{\delta W_{1x_1} W_{1x_2}}{\delta t} = -2\rho q I_1(s) f(Re_0)(\vec{X}, \operatorname{grad} W_{0x_1} W_{0x_2}) - $$

$$- 3q\rho I_2(s)\, \overline{u_{x_1} u_{x_2}}. \tag{13}$$

Correlations (12) and (13) provide all that is necessary to effect transition to a description of a discrete medium from the standpoint of continuum properties.

III. EQUATIONS OF TRANSFER

1. Consider a system of physical points located in a

continuous velocity field. Let these points be given their own velocity of a disharmonious motion, superimposed upon the given continuous velocity field. In such a case the velocity of a physical point relative to a stationary system of coordinates can be expressed as being

$$\vec{W} = \vec{W_o} + \vec{u}$$

Velocity $\vec{W_o}$, as well as mean velocity of thermal motion, generally speaking, is a function of coordinates and time. The indicated continuous velocity field may be represented as the motion of a hypothetical continuous medium which carries along the physical points which are in a state of disharmonious motion. This picture is akin to the motion of a fluid in which Brownian particles are distributed.

2. Each gas molecule in motion transfers with it a certain amount of mass, kinetic energy, mechanic moment, etc. Let us denote, as was done previously, the quantitative expression of any of these properties by means of Q. This quantity, generally speaking, is a function of time and a component of the total velocity of the molecules. The amount Q expanded to all molecules contained in the unit of volume, and averaged within that volume, will be denoted by $\overline{NQ}$.

The total alteration with the unit of time of the quantity $\overline{NQ}$, contained in the unit of volume, will be equal to

$$\frac{d\overline{NQ}}{dt} \tag{14a}$$

A change of the quantity $\overline{NQ}$ may take place also by the action of external forces capable of altering the velocity of the molecules.

Since N is not dependent upon velocities, change of $\overline{NQ}$ per unit of time as a result of the action of external forces can be written in the following form

$$N\left[\frac{\partial \bar{Q}}{\partial W_{x_1}} F_{x_1} + \frac{\partial \bar{Q}}{\partial W_{x_2}} F_{x_2} + \frac{\partial \bar{Q}}{\partial W_{x_3}} F_{x_3}\right]. \qquad (14b)$$

Wherein components of accelerations are replaced by components of the external force.

If $\vec{K}$ is used to denote the vector, the components of which along the coordinate axes are given by quantities of the form

$$\frac{\partial \bar{Q}}{\partial W_{x_1}}, \quad \frac{\partial \bar{Q}}{\partial W_{x_2}}, \quad \frac{\partial \bar{Q}}{\partial W_{x_3}},$$

then the foregoing expression can be given as the scalar product

$$N(\vec{K}, \vec{F}). \qquad (14c)$$

In addition to the indicated changes, there may be changes of the quantity $\overline{NQ}$ as a result of the fact that the number of molecules within the element of volume may undergo alteration -- molecules may enter into the element of volume and leave it.

Because of this, changes of the quantity $\overline{NQ}$ per unit of time may be written as follows

$$\overline{NQ}\,\mathrm{div}\,\overrightarrow{W_0} + \mathrm{div}\,\overline{\vec{u}QN}. \qquad (14\,d)$$

Here the expression div $\overline{\vec{u}QN}$ has a purely symbolic meaning, since it must be always remembered that before effecting this operation, the following quantities must be worked out

$$\overline{u_{x_1}QN}, \quad \overline{n_{x_2}QN}, \quad \overline{u_{x_3}QN}.$$

3. The sum total of all possible changes per unit of time of the quantity $\overline{NQ}$ can be written as being

$$\frac{d\overline{NQ}}{dt} + \overline{NQ}\,\mathrm{div}\,\overrightarrow{W_0} + \mathrm{div}\,\overline{\vec{u}NQ} = N\frac{\delta\overline{Q}}{\delta t} + N(\vec{K},\vec{F}).$$

wherein the quantity

$$N\,\frac{\delta\overline{Q}}{\delta t}$$

denotes the change, per unit of time, of the quantity Q on pair-intercollisions.

Let us assume in this equation $\overline{Q} = m$, and denote [redacted] mN by means of ρ, the density of the medium. In such a case equation (15) can be written as follows:

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}\,\overrightarrow{W_0} = 0 \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \mathrm{div}\,\rho\overrightarrow{W_0} = 0. \qquad (15\,a)$$

Equations (15a) are known as continuity equations. They can be used to simplify equation (15). Indeed the value of derivative

$$\frac{d\overline{QN}}{dt}$$

can be expressed as follows

$$\frac{d\overline{QN}}{dt} = N\frac{d\overline{Q}}{dt} + \overline{Q}\frac{dN}{dt}$$

On combining ~~thus equation~~ this identity with one of the equations (15a), the differential equation (15) can be transformed into the following

$$N\frac{d\overline{Q}}{dt} + div\,\overline{\vec{u}QN} = N\frac{\delta Q}{\delta t} + N(\vec{K},\vec{F}). \qquad (15b)$$

4. We ~~Assuming~~ assume, in equation (15b), Q to be consecutively equal to

$$mW_{x_1} = m(W_{0x_1} + u_{x_1}),\ mW_{x_3} = m(W_{x_2} + u_{x_2}),\ mW_{x_3} = m(W_{x_3} + u_{x_3}).$$

As a result we have, if it be assumed that collisions of molecules do not change the ~~quantity of motion~~ total momentum and if the product mN is replaced by means of density ρ:

$$\rho\frac{dW_{0x_1}}{dt} + \frac{\partial\rho\overline{u^2_{x_1}}}{\partial x_1} + \frac{\partial\rho\overline{u_{x_2}u_{x_3}}}{\partial x} + \frac{\partial\rho\overline{u_{x_1}u_{x_3}}}{\partial x_3} = \rho F_{x_1},$$

$$\rho\frac{dW_{0x_2}}{dt} + \frac{\partial\rho\overline{u_{x_1}u_{x_3}}}{\partial x_1} + \frac{\partial\rho\overline{u^2_{x_2}}}{\partial x_2} + \frac{\partial\rho\overline{u_{x_2}u_{x_3}}}{\partial x_3} = \rho F_{x_2}, \qquad (16)$$

$$\rho \frac{dW_{0x_3}}{dt} + \frac{\partial \rho u_{x_1} u_{x_3}}{\partial x_1} + \frac{\partial \rho u_{x_2} u_{x_3}}{\partial x_2} + \frac{\partial \rho u_{x_3}^2}{\partial x_3} = \rho F_{x_3}.$$

Equations (16) are the initial correlations for a derivation of equations of hydrodynamics of ideal and viscous fluid.

5. In the case where thermal motion of molecules is not disrupted by the apparent motion of the medium, the following equations hold

$$u_{x_1} u_{x_2} = u_{x_1} u_{x_3} = u_{x_2} u_{x_3} = 0, \qquad u_{x_1}^2 = u_{x_2}^2 = u_{x_3}^2 \qquad (17)$$

For such a motion of the medium the following assertion is also true: pressure within the gas is equal in all directions, normal to any surface therein, and is expressed as

$$p = \frac{1}{3}\rho\left(\bar{u}_{x_1}^2 + \bar{u}_{x_2}^2 + \bar{u}_{x_3}^2\right) = \rho \bar{u}_{x_1}^2 = \rho \bar{u}_{x_2}^2 = \rho \bar{u}_{x_3}^2. \qquad (17a)$$

In this instance equations (16) assume the following vectorial form

$$\rho \frac{d\vec{W_0}}{dt} + \operatorname{grad} p = \rho \vec{F}. \qquad (17b)$$

For a non-turbulent motion, that is, when the following equation holds

$$\operatorname{curl} \vec{W_0} = 0,$$

the equation (17) can be formulated, as is known, in the following form

$$\rho \frac{\partial \vec{W_0}}{\partial t} + \rho \, grad \frac{W_0^2}{2} + grad \, p = \rho \vec{F}. \qquad (17c)$$

Equations (17b) and (17c) are known ones; they are Euler's equations for an ideal fluid. From the given derivation of these equations it is fully apparent what must be understood as being an ideal fluid and when these equations can be utilized for the solution of ideal problems.

IV. CALCULATION OF FUNCTIONS $\overline{\rho u^2_{x_1}}, \overline{\rho u^2_{x_2}}, \overline{\rho u^2_{x_3}}$

1. Let us substitute in equation (15b) in lieu of Q the quantity W_{x_1}; then in view of the fact that

$$\overline{Q} = W_{0x_1} \quad and \quad \frac{\partial \overline{Q}}{\partial W_{0x_1}} = 1,$$

we have,

$$N \frac{dW_{0x_1}}{dt} + div \overline{Nuu_{x_1}} = NF_{x_1}.$$

Of such equations we shall have three corresponding to the components of velocity $\vec{W_0}$. From these equations we can derive one of the following form

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$$N\left(\vec{K}, \frac{d\vec{W_0}}{dt}\right) + \left\{ \frac{\partial \bar{Q}}{\partial W_{0x_1}} \operatorname{div} \overline{N\vec{u}u_{x_1}} + \frac{\partial \bar{Q}}{\partial W_{0x_2}} \operatorname{div} \overline{N\vec{u}u_{x_2}} + \right.$$

$$\left. + \frac{\partial \bar{Q}}{\partial W_{0x_3}} \operatorname{div} \overline{N\vec{u}u_{x_3}} \right\} = N(\vec{K}, \vec{F}). \qquad (18)$$

Let us introduce into equations (15b) and (18) in lieu of $\bar{Q}$ the quantity $mW^2_{1x_1}$ and subtract the second from the first; we have then

$$\rho \frac{\delta \overline{W^2_{1x_1}}}{\delta t} = \rho \frac{d\overline{u^2_{x_1}}}{dt} + \frac{\partial(\rho \overline{u^3_{x_1}})}{\partial x_1} + \frac{\partial(\rho \overline{u^2_{x_1} u_{x_2}})}{\partial x_2} + \frac{\partial(\rho \overline{u^2_{x_1} u_{x_3}})}{\partial x_3} +$$

$$+ 2\rho \left(\overline{u^2_{x_1}} \frac{\partial W_{0x_1}}{\partial x_1} + \overline{u_{x_1} u_{x_2}} \frac{\partial W_{0x_1}}{\partial x_2} + \overline{u_{x_1} u_{x_3}} \frac{\partial W_{0x_1}}{\partial x_3} \right). \qquad (19)$$

In this equation the members

$$\frac{\partial \rho \overline{u^3_{x_1}}}{\partial x_1}, \quad \frac{\partial \rho \overline{u^2_{x_1} u_{x_2}}}{\partial x_3}, \quad 2\rho \overline{u_{x_1} u_{x_2}} \frac{\partial W_{0x_1}}{\partial x_2}, \quad 2\rho \overline{u_{x_1} u_{x_3}} \frac{\partial W_{0x_1}}{\partial x_3}$$

are very small quantities -- of the same order as $\overline{u_{x_1} u_{x_2}}$, $\overline{u^2_{x_1}} - \overline{u^2_{x_2}}$ and so forth; the member appearing in the left hand portion of this equation cannot be considered as being very small, since it is also equal to the right hand portion of equation (12) which consists of products composed of factors having small and large values. The quantity q must be considered as a factor having a large value. This follows from a comparison of the expression for the coefficient of internal friction, ensuing from Maxwell's theory with the number obtained on experimental determination of this coefficient.

Taking into account all that has been stated herein, we can obtain, on comparison of equations (12) and (19), the following expression

$$\rho \frac{\overline{du^2_{x_1}}}{dt} + 2\rho \overline{u^2_{x_1}} \frac{\partial W_{0x_1}}{\partial x_1} = -2\rho q^2 I_1(s) f(Re_0)(\vec{X}, \operatorname{grad} W^2_{0x_1}) + $$
$$+ 3q\rho I_2(s)[p - \rho \bar{u}^2_{x_1}]. \qquad (20)$$

2. Before evolving from equation (20) the value $\rho \overline{u}^2_{x_1}$, we will transform this equation into a different, more convenient form. From determination of pressure p, it follows that

$$\frac{dp}{dt} = \frac{\overline{u}^2}{3}\frac{d\rho}{dt} + \frac{1}{3}\rho \frac{d\overline{u}^2}{dt}.$$

Connecting therewith the continuity equation, we have

$$\frac{dp}{dt} + p \operatorname{div} \vec{W_0} = \frac{\rho}{3}\frac{d\overline{u}^2}{dt} \approx \rho \frac{d\overline{u^2_{x_1}}}{dt}. \qquad (21)$$

Substituting these correlations into equation (20) we have

$$\frac{dp}{dt} + p \operatorname{div} \vec{W_0} + 2\rho \overline{u}^2_{x_1} \frac{\partial W_{0x_3}}{\partial x_1} = -2\rho^2 q I_1(s) f(Re_0)(\vec{X}, \operatorname{grad} W^2_{0x_1}) + $$
$$+ 3\rho q I_2(s)[p - \rho \bar{u}^2_{x_1}]. \qquad (22a)$$

From this the following approximate value for the quantity $\rho \overline{u}^2_{x_1}$ can be obtained

$$\rho \overline{u}^2_{x_1} = p - \frac{\rho I_1(s) f(Re_0)}{3I_2(s)} (\vec{X}, \operatorname{grad} W^2_{0x_1}) - \frac{2p}{3\rho q I_2(s)} \left[\frac{\partial W_{0x_1}}{\partial x_1} + \frac{1}{2} \operatorname{div} \vec{W_0} \right] -$$

$$- \frac{p}{3\rho q I_2(s)} \frac{d \log p}{dt}. \qquad (22b)$$

If the flow of fluid is fully adiabatic, then the following equation holds for such a case

$$\frac{d \log p}{dt} - \gamma \frac{d \log \rho}{dt} = \frac{d \log p}{dt} + \gamma \operatorname{div} \vec{W} = 0.$$

By means of this equation we can formulate the correlation (22b) as follows

$$\rho \overline{u}^2_{x_1} = p - \frac{2\rho I_1(s) f(Re_0)}{3I_2(s)} (\vec{X}, \operatorname{grad} W^2_{0x_i}) - \frac{2p}{3\rho q I_2(s)} \left[\frac{\partial W_{0x_i}}{\partial x_i} - \frac{\gamma - 1}{2} \operatorname{div} \vec{W_0} \right]. \qquad (23)$$

V. CALCULATION OF FUNCTIONS: $\rho \overline{u_{x_1} u_{x_2}}, \rho \overline{u_{x_1} u_{x_3}}, \rho \overline{u_{x_2} u_{x_3}}$

1. Let us introduce into the transfer equation (15b) in place of Q a quantity equal to $W_{1x_1} W_{1x_2}$, and by means of equation (15) let us eliminate from it the external force; then we have

$$\rho \frac{\delta \overline{W_{1x_1} W_{1x_2}}}{\delta t} = \rho \frac{d \overline{u_{x_1} u_{x_2}}}{dt} + \left[\frac{\partial (\rho \overline{u^2_{x_1} u_{x_2}})}{dt} + \frac{\partial (\rho \overline{u_{x_1} u_{x_2} u_{x_3}})}{\partial x_3} \right] +$$

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$$+\rho\left[\bar{u}_{x_1}\frac{\partial W_{0x_2}}{dx_1}+\bar{u}^2_{x_2}\frac{\partial W_{0x_1}}{\partial x_2}+\overline{u_{x_2}u_{x_3}}\frac{\partial W_{0x_1}}{\partial x_3}+\overline{u_{x_1}u_{x_3}}\frac{\partial W_{0x_2}}{\partial x_3}+\right.$$

$$\left.+\overline{u_{x_1}u_{x_2}}\frac{\partial W_{0x_1}}{\partial x_1}+\overline{u_{x_1}u_{x_2}}\frac{\partial W_{0x_2}}{\partial x_2}\right]. \qquad (24)$$

The expression so obtained, as well as the expression (19), admits simplifications; we can write

$$\rho\frac{\delta W_{1x_1}W_{1x_2}}{\delta t}=\rho\left[\bar{u}^2_{x_2}\frac{\partial W_{0x_1}}{\partial x_1}+\bar{u}^2_{x_2}\frac{\partial W_{0x_1}}{\partial x_2}\right]$$

or, approximately,

$$\rho\frac{\delta\overline{W_{1x_1}W_{1x_2}}}{\delta t}=\rho\left[\frac{\partial W_{0x_1}}{\partial x_2}+\frac{\partial W_{0x_3}}{\partial x_1}\right]. \qquad (24a)$$

Comparing equations (19) and (24a) we have

$$\rho\left[\frac{\partial W_{0x_1}}{\partial x_2}+\frac{\partial W_{0x_2}}{\partial x_1}\right]=-2\rho^2 qI_1(s)f(Re_0)(\vec{\bar{X}},\text{grad } W_{0x_1}W_{0x_2})-$$

$$-3\rho^2 qI_2(s)\overline{u_{x_1}u_{x_2}};$$

wherefrom we have

$$\rho\overline{u_{x_1}u_{x_2}}=-\frac{2\rho I_1(s)f(Re_0)}{3I_2(s)}(\bar{X},\text{grad } W_{0x_1}W_{0x_2})-\frac{\rho}{3\rho qI_2(s)}\left[\frac{\partial W_{0x_1}}{\partial x_2}+\frac{\partial W_{0x_2}}{\partial x_1}\right]. \qquad (25)$$

The correlations for quantities $\rho\,\overline{U_{x1}\text{ and }U_{x2}}$ and $\rho\overline{U_{x2}\,U_{x3}}$ can be written in an analogous manner.

Thus equations (23) and (25) complete the solution of the proposed problem. To obtain the hydrodynamics equation it is necessary merely to insert these correlations into equation (6).

VI. PARTICULARIZATION OF HYDRODYNAMICS EQUATION

1. Let us transform the first equation of the equation system (6) by means of correlations of the type of (23) and (25). As a result we have

$$\frac{dW_{0x_1}}{dt} - \frac{2I_1(s)f(Re_0)}{3I_2(s)}\left[\frac{\partial W^2_{0x_1}}{\partial x_1} + \frac{\partial W_{0x_1}W_{0x_2}}{\partial x_2} + \frac{\partial W_{0x_1}W_{0x_3}}{\partial x_3}\right] =$$

$$= F_{x_1} - \frac{1}{\rho}\frac{\partial p}{\partial x_1} + \frac{\eta}{\rho}\Delta_2 W_{0x_1} + \frac{(2-\gamma)\eta}{\rho}\frac{\partial}{\partial x_1}\,\mathrm{div}\,\vec{W_0}.$$

Here η denotes the quantity

$$\eta = \frac{p}{3g\rho I_2(s)}.$$

The expression within the brackets of the equation thus obtained we can formulate as follows

$$\frac{\partial W^2_{0x_1}}{\partial x_1} + \frac{\partial W_{0x_1}W_{0x_2}}{\partial x_2} + \frac{\partial W_{0x_1}W_{0x_3}}{\partial x_3} = \frac{1}{2}\frac{\partial W_0^2}{\partial x_1} + W_{0x_1}\,\mathrm{div}\,\vec{W_0} +$$

$$+ 2(W_{0x_3}\omega_2 - W_{0x_2}\omega_3).$$

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Here $\vec{\omega}$ denotes curl $\vec{W_0}$.

Upon discarding the index (0) and writing the above derived equation in vectorial form we have

$$\frac{d\vec{W}}{dt}\,\frac{2I_1(s)f(Re_0)}{3I_2(s)}\left[\frac{1}{2}\,\mathrm{grad}\,W^2+\vec{W}\,\mathrm{div}\,\vec{W}+2\left[\vec{\omega},\vec{W}\right]\right]=$$

$$=\vec{F}-\frac{1}{\rho}\,\mathrm{grad}\,p+\frac{\eta}{\rho}\Delta_2\vec{W}+\frac{(2-\gamma)\eta}{\rho}\,\mathrm{grad}\,\mathrm{div}\,\vec{W}; \qquad (26)$$

on the other hand we have

$$\frac{d\vec{W}}{dt}=\frac{\partial\vec{W}}{\partial t}+\frac{1}{2}\,\mathrm{grad}\,W^2+2\left[\vec{\omega},\vec{W}\right],$$

therefore equation (26) can also be written as follows

$$\frac{\partial\overleftarrow{W}}{\partial t}+\frac{1}{2}(1-\beta)\,\mathrm{grad}\,W^2-\beta\,\vec{W}\,\mathrm{div}\,\vec{W}+2(1-\beta)\left[\vec{\omega},\vec{W}\right]=$$

$$=\vec{F}-\frac{1}{\rho}\,\mathrm{grad}\,p+\frac{\eta}{\rho}\Delta_2\vec{W}+\frac{(2-\gamma^{\eta})}{\rho}\,\mathrm{grad}\,\mathrm{div}\,\vec{W}. \qquad (26a)$$

In this equation β is used to denote the quantity

$$\beta=\frac{2I_1(s)f(Re_0)}{3I_2(s)} \qquad (26b)$$

2. Equation (26a) is of a more general nature than the equation of Navier-Stokes, which Maxwell strived to substantiate from the standpoint of the molecular kinetic theory. Indeed, if it is assumed that $\beta = 0$ and $\gamma = \frac{5}{3}$, which corresponds to a monoatomic gas, then equation (26a) is transformed exactly into the Navier-Stokes equation. But as we have seen, quantity β can become zero only in the case of the transfer velocities of colliding molecules being equal, which does not always obtain.

In the course of his computations, striving to take into account quantities of the first order only, Maxwell did not notice that his limitations contained in a concealed form adiabatic equations of a monoatomic gas; hence in lieu of the value $(2 - \gamma)$ he obtained a value equal to 1/3.

Let us consider an interesting result of equation (26a). Let the external forces have no effect upon the medium, and let its flow be such that we can disregard in equation (26a) the viscosity members and members making allowance for compressibility. We will consider the motion as being potential and steady. In this case the following simple equation is derived.

$$\frac{1-\beta}{2} \operatorname{grad} W^2 = -\frac{1}{\rho} \operatorname{grad} p,$$

wherefrom we have

$$\frac{1-\beta}{2} W^2 = -\int \frac{dp}{\rho} + const.$$

If the motion follows the adiabatic law, this equation can be transformed to give

$$\frac{1-\beta}{2} W^2 = -\frac{A\gamma}{(\gamma-1)} \left[\rho^{\gamma-1} - \rho_0^{\ \gamma-1} \right]$$

Where A is used to denote the adiabatic constant.

Resolving this equation for ρ , we have

$$\rho = \rho_0 \left[1 - \frac{(1-\beta)(\gamma-1)}{A\rho_0^{\ \gamma-1}} W^2 \right]^{\frac{1}{\gamma-1}} \qquad (27)$$

From the thus obtained equation it is apparent that two flows exist which differ sharply from one another: (1) when $\beta < 1$, formula (27) shows that with increasing velocity the density of the medium decreases; these are flows of Euler; (2) when $\beta > 1$, it follows from formula (27) that with increasing velocity the density increases.

4. In the theory of gaseous currents of S. A. Chaplygin a two-dimensional problem was studied. For a constant condition in this case the continuity equation is

$$\frac{\partial(\rho W_{x_1})}{\partial x_1} + \frac{\partial(\rho W_{x_2})}{\partial x_2} = 0$$

If the flow is potential, then the above equation indicates the existence of a function ψ determined by correlations

$$\frac{\rho}{\rho_0} W_{x_1} = \frac{\partial \psi}{\partial x_2} \quad and \quad \frac{\rho}{\rho_0} W_{x_2} = -\frac{\partial \psi}{\partial x_1} .$$

Denoting by φ the potential of velocities and bearing in mind equation (27), the above-given correlations can be presented in the following form

$$\frac{\partial\psi}{\partial x_2} = \left[1 - \frac{(1-\beta)W^2}{K}\right]^S \frac{\partial\varphi}{\partial x_1}, \quad \frac{\partial\psi}{\partial x_1} = \left[1 - \frac{(1-\beta)W^2}{K}\right]^S \frac{\partial\varphi}{\partial x_2}.$$

If in accordance with S. A. Chaplygin we denote by τ the ratio $\frac{W^2}{K}$, the equations so obtained can be written as follows

$$\frac{\partial\psi}{\partial x_2} = \left[1-(1-\beta)\tau\right]^S \frac{\partial\varphi}{\partial x_1}, \quad \frac{\partial\psi}{\partial x_1} = \left[1-(1-\beta)\tau\right]^S \frac{\partial\varphi}{\partial x_2}. \qquad (28)$$

When $\beta < 1$, equations (28) are reduced exactly to those studied by S. A. Chaplygin. When $\beta > 1$, the equations of S. A. Chaplygin acquire the opposite sign in the variable τ. Indeed, we have

$$\frac{\partial\psi}{\partial x_2} = \left[1-\tau_1\right]^S \frac{\partial\varphi}{\partial x_1}, \qquad \frac{\partial\psi}{\partial x_1} = \left[1-\tau_1\right]^S \frac{\partial\varphi}{\partial x_2}$$

$$\tau_1 = (1-\beta)\tau, \qquad \beta < 1; \qquad (28a)$$

$$\frac{\partial\psi}{\partial x_2} = \left[1+\tau_1\right]^S \frac{\partial\varphi}{\partial x_1}, \qquad \frac{\partial\psi}{\partial x_1} = \left[1+\tau_1\right]^S \frac{\partial\varphi}{\partial x_2}$$

$$\tau_1 = (\beta-1)\tau, \qquad \beta > 1. \qquad (28b)$$

Transforming these equations can be done according to the procedure proposed by S. A. Chaplygin, that is, by selecting as independent variables τ_1 and θ -- the angle formed by flow velocity with axis x_1.

In this instance equations (28a) are converted into equations of the following form

$$\frac{\partial}{\partial \tau_1}\left[2\tau_1(1-\tau_1)^{-s}\frac{\partial \psi}{\partial \tau_1}\right]+\frac{1-(1+2s)\tau_1}{2\tau_1(1-\tau_1)}(1-\tau_1)^{-s}\frac{\partial^2 \psi}{\partial \theta^2}=0. \quad (29a)$$

Equations (28b) are converted to the form

$$\frac{\partial}{\partial \tau_1}\left[2\tau_1(1+\tau_1)^{-s}\frac{\partial \psi}{\partial \tau_1}\right]+\frac{1+(1+2s)\tau_1}{2\tau_1(1+\tau_1)}(1+\tau_1)^{-s}\frac{\partial^2 \psi}{\partial \theta^2}=0. \quad (29b)$$

Thus the entire theory of gaseous streams for all velocities must be encompassed by an analysis of equations (29a) and (29b). However, even without solving these equations some conclusions can be reached relative to the behavior of gaseous streams on the flow of gas through a slit, on the basis of physical considerations. Discharge of gas through the slit will increase with increasing pressure of gas egress. The magnitude of this discharge will follow laws which ensue from the integral of equation (29a).

The presented concept raised the question of revising some of the postulates of high velocity gas dynamics.

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